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Soliton and breather states of the quantum sine-Gordon model in light cone coordinates through the exact QIST method

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Abstract. The quantum sine-Gordon equation in light cone coordinates is solved exactly through the quantum inverse scattering technique by finding a suitable gauge in which the model becomes ultralocal. The existence of breather bound states is shown and a consistent soliton representation is produced. Physical consequences are discussed along with a possible gauge invariance of the system at the quantum level.

1. Introduction

After the overwhelming success of the inverse scattering transform (IST) for integrating exactly a class of non-linear field equations in one space dimension [1, 2] its quantum version (QIST) was developed [3-5] to solve the corresponding quantum models. But unfortunately, while the QIST gave excellent result for models like the non-linear Schrödinger equation (NLS) [6, 7] with ultralocality, it was found to be unsuccessful in treating models like KdV, derivative NLS [8], non-linear σ models [9], etc, with non-ultralocality. The sine-Gordon (SG) model is also an example of non-ultralocal class. The brilliance of Sklyanin *et al* [10], however, was able to tackle SG through QIST in a particular gauge and find breather solutions. A recent work [11] also extended the earlier approach for the SG model to an elegant continuum limit. In spite of their success both the above methods [10, 11] were confined only to the laboratory system of coordinates (LBC) and failed to find (anti-)soliton states. Contrary to QIST, in the classical case it is easier to solve SG through IST in light cone coordinates (LCC) and obtain both (anti-)soliton and breather solutions [1].

It is, therefore, an open question whether it is possible to pursue QIST for SG in LCC and whether soliton solutions can be found for this model. Our aim is to answer these questions. We overcome the difficulties for solving SG in LCC through QIST by finding a suitable gauge, in which the model becomes ultralocal and leads to an exact solution. An added advantage of our method is that potentially it has wide applications in dealing with models associated with the AKNS spectral problem including non-relativistic models like quantum MKdV [12]. An attempt has also been made to construct a consistent representation of bosonic creation operators leading to (anti-)soliton states with a spectrum similar to that found semiclassically [13]. The bosonic particle states formed through interaction of a soliton and an anti-soliton in their turn may also form a bound-state breather solution. The classical gauge equivalent Lax operator representation for a particular model breaks, in general, in the quantum case. Therefore, QIST applied to any system is generally valid for a particular gauge choice. We, however, have demonstrated that our solution of SG in the light cone exhibits gauge invariance even at the quantum level.

In § 2 we introduce the model and explicitly construct the R matrix starting from a suitable Lax operator. Section 3 gives the N-particle eigenstates and the corresponding eigenvalues. The breather bound state and the possible soliton state are also constructed in this section. Section 4 discusses the gauge equivalence of the model at the quantum level and § 5 is a concluding one.

2. The quantum sine-Gordon model

The sine-Gordon model in LCC $x=\frac{1}{2}(x^1-x^0)$, $t=\frac{1}{2}(x^1+x^0)$ is given by the equation

$$\theta_{xt}=m^2\sin\theta. \tag{1}$$

The linear system

$$\partial_x T_y^x(\lambda)=L(x,\lambda)T_y^x(\lambda) \tag{2}$$

associated with this model is usually given by the AKNS type Lax operator representation

$$L(x,\lambda)=\mathrm{i}(\tfrac{1}{2}m\lambda\sigma_3-\tfrac{1}{2}\partial_x\theta(x)\sigma_1). \tag{3}$$

However, one finds that the Lax operator (3) leads to a non-ultralocal Poisson bracket structure which forbids direct application of QIST to the corresponding quantum case. Therefore, in such cases one has to look for some suitable Lax operator of this model which is free from non-ultralocality. Fortunately, we are able to obtain the following ultralocal Lax operator given as

$$L(x,\lambda)=\mathrm{i}\left[\frac{\pi(x)}{4}\sigma_1+\frac{m\lambda}{2}\cos\left(\frac{q(x)}{2}\right)\sigma_3+\frac{m\lambda}{2}\sin\left(\frac{q(x)}{2}\right)\sigma_2\right] \tag{4}$$

by introducing new variables in the form $\partial_x\theta(x)=\frac{1}{2}(\partial_x q(x)-\pi(x))$, where $q(x)$ and $\pi(x)$ satisfy the canonical relations

$$[q(x),\pi(y)]=4\hbar\beta^2\delta(x-y) \tag{5a}$$

and

$$[q(x),q(y)]=[\pi(x),\pi(y)]=0. \tag{5b}$$

Note that the above properties (5a) and (5b) are consistent with $[\partial_x\theta(x),\partial_y\theta(y)]=-2\hbar\beta^2\partial_x\delta(x-y)$ which is derivable from the standard equal-time ($x^0=y^0=0$) commutation relation $[\theta(y^1),\partial_{x^0}\theta(x^1)]=\hbar\beta^2\delta(x^1-y^1)$. The time evolution operator in this case is given by

$$M=\tfrac{1}{4}\mathrm{i}\{\partial_t q\sigma_1-(2m/\lambda)[\cos(\theta-\tfrac{1}{2}q)\sigma_3-\sin(\theta-\tfrac{1}{2}q)\sigma_2]\}. \tag{6}$$

One may easily check that the compatibility condition of (4) and (6), $\partial_t L-\partial_x M+[L,M]=0$, leads to the same sine-Gordon equation (1) in the θ variable.

Therefore, for solving equation (1) through QIST we may start from the Lax operator (4) considering the fields as quantum operators and express the monodromy matrix as the path-ordered product:

$$T_y^x(\lambda)=P\exp\left(\int_y^x \mathrm{d}\xi\, L(\xi,\lambda)\right). \tag{7}$$

The methods developed in [10, 11] may be applied now with no difficulty. We present here in brief the main steps involved; the details can be found in [11]. The most important relation in QIST is the Yang-Baxter equation

$$R(\lambda, \mu) T_y^x(\lambda) \otimes T_y^x(\mu) = T_y^x(\mu) \otimes T_y^x(\lambda) R(\lambda, \mu) \tag{8}$$

which guarantees the complete integrability of the system and, consequently, the major aim of this approach is to find the related R matrix. For this, we first have to obtain an equation for the direct product of monodromy matrices. However, this procedure involves the product of local field operators at the same point, giving rise to singularities, which should be carefully taken into account. We have

$$\partial_x (T_y^x(\lambda) \otimes T_y^x(\mu)) = \Gamma(x, \lambda, \mu) T_y^x(\lambda) \otimes T_y^x(\mu) \tag{9}$$

with

$$\Gamma(x, \lambda, \mu) = L(x, \lambda) \otimes \mathbb{1} + \mathbb{1} \otimes L(x, \mu) + \int_{x-\Delta}^{x} \mathrm{d}\xi [L(x, \lambda) \otimes L(\xi, \mu) + L(\xi, \lambda) \otimes L(x, \mu)] + \mathrm{O}(\Delta^2). \tag{10}$$

Considering now the singularities of the operator products in (10) we observe that [11] the leading singularities are of the order $1/\Delta$ only, provided $\hbar\beta^2 < 2\pi$. A typical singularity is obtained in the form

$$[\pi^{(-)}(x), :\exp[(\mathrm{i}/2) q(y)]:] = 2\mathrm{i}\hbar \delta^{(-)}(x-y) :\exp[(\mathrm{i}/2) q(y)]: \tag{11}$$

$$q(x) = q^{(+)}(x) + q^{(-)}(x)$$

where $\pi^{(-)}(x)$ and $q^{(-)}(y)$ are the 'negative frequency' components. Similarly, other relevant terms can also be obtained leading finally, in the limit $\Delta \to 0$, to the expression

$$\Gamma(x, \lambda, \mu) = L(x, \lambda) \otimes \mathbb{1} + \mathbb{1} \otimes L(x, \mu) - \tfrac{1}{2} m\gamma [\lambda(\sigma_3 \otimes \sigma_1 + \sigma_2 \otimes \sigma_1) - \mu(\sigma_1 \otimes \sigma_2 + \sigma_1 \otimes \sigma_3)] \tag{12}$$

with $\gamma = \hbar\beta^2/2$. Note that the terms with γ appear as a consequence of the normal ordering procedure described above. Due to (9) relation (8) reduces now to

$$R(\lambda, \mu) \Gamma(x, \lambda, \mu) = \Gamma(x, \mu, \lambda) R(\lambda, \mu) \tag{13}$$

which yields

$$R(\lambda, \mu) = \mathbb{1} \otimes \mathbb{1} + f(\lambda, \mu) \sigma_3 \otimes \sigma_3 + g(\lambda, \mu) \sigma_2 \otimes \sigma_2 + h(\lambda, \mu) \sigma_1 \otimes \sigma_1 \tag{14}$$

with

$$f(\lambda, \mu) = g(\lambda, \mu) = \left(\frac{1+\gamma^2}{1-\gamma^2}\right)\left(1 - \frac{2\mathrm{i}\gamma}{1-\gamma^2} \frac{\lambda+\mu}{\lambda-\mu}\right)^{-1}$$

and

$$h(\lambda, \mu) = \left(1 - \frac{2\mathrm{i}\gamma}{1-\gamma^2} \frac{\lambda-\mu}{\lambda+\mu}\right)\left(1 - \frac{2\mathrm{i}\gamma}{1-\gamma^2} \frac{\lambda+\mu}{\lambda-\mu}\right)^{-1}.$$

In the classical limit (14) reduces clearly to the classical r matrix [5]

$$R \approx \mathbb{P} + \mathrm{i}\gamma \mathbb{P} r \qquad \gamma \to 0$$

where

$$\mathbb{P} = \tfrac{1}{2}(\mathbb{1} \otimes \mathbb{1} + \sigma^a \otimes \sigma^a)$$

and

$$r = a(\lambda, \mu)\sigma_1 \otimes \sigma_1 + b(\lambda, \mu)(\sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) \tag{15}$$

with

$$a(\lambda, \mu) = \frac{1}{4}\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2} \qquad b(\lambda, \mu) = \frac{1}{2}\frac{\lambda\mu}{\lambda^2-\mu^2}.$$

It is interesting to note that the quantum R matrix (14) obtained here for SG in LCC coincides exactly with that found in LBC [11] if the parametrisation

$$K^0 = \tfrac{1}{2}m(\lambda + 1/\lambda) \qquad K^1 = \tfrac{1}{2}m(\lambda - 1/\lambda) \tag{16}$$

is introduced and γ is replaced by $\gamma/4$, which reflects the difference in the normal ordering prescription in these cases.

Switching over to the infinite interval limit we may define now the scattering data matrix $T(\lambda)$ as

$$T(\lambda) = \lim_{\substack{x\to+\infty \\ y\to-\infty}} \exp(-\tfrac{1}{2}\mathrm{i}m\lambda\sigma_3 x) T_y^x(\lambda) \exp(\tfrac{1}{2}\mathrm{i}m\lambda\sigma_3 y) \tag{17a}$$

which may be expressed explicitly in the form

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ -\bar{B}(\lambda) & \bar{A}(\lambda) \end{pmatrix}. \tag{17b}$$

Following the same procedure as developed in [11, 14] it may be shown that the Yang-Baxter relation (8) in the infinite interval limit reduces to

$$R_+(\lambda, \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R_-(\lambda, \mu) \tag{18}$$

where

$$R_\pm(\lambda, \mu) = \lim_{x\to\infty} \begin{pmatrix} (1+f) & 0 & 0 & (h-f)\exp[\mp\mathrm{i}m(\lambda+\mu)x] \\ 0 & (l-f)\exp[\mp\mathrm{i}m(\lambda-\mu)x] & (h+f) & 0 \\ 0 & (h+f) & (l-f)\exp[\pm\mathrm{i}m(\lambda-\mu)x] & 0 \\ (h-f)\exp[\pm\mathrm{i}m(\lambda+\mu)x] & 0 & 0 & (1+f) \end{pmatrix}. \tag{19}$$

Now one may easily derive a commutation algebra between the elements of the scattering data matrix (17b) from (18) and (19) by taking properly the limit $x\to+\infty$ [11, 14]. In this procedure one encounters singular terms, which should be understood in the principal value sense $\lim_{x\to\pm\infty} P\exp[\mathrm{i}\lambda x]/\lambda = \pm\mathrm{i}\pi\delta(\lambda)$ resulting finally in the relations

$$[A(\lambda), A(\mu)] = [A(\lambda), \bar{A}(\mu)] = 0 \tag{20a}$$

$$[B(\lambda), \bar{B}(\mu)] = -2\pi\gamma(\lambda+\mu)\delta(\lambda-\mu)\bar{A}(\lambda)A(\lambda) \tag{20b}$$

$$A(\lambda)\bar{B}(\mu) = \left(\frac{h(\mu,\lambda)+f(\mu,\lambda)}{1+f(\mu,\lambda)} - \pi\gamma(\lambda-\mu)\delta(\lambda+\mu)\right)\bar{B}(\mu)A(\lambda) \tag{20c}$$

etc, where the symmetry relations $\bar{A}(\lambda) = A(-\lambda)$ and $\bar{B}(\lambda) = B(-\lambda)$ for real λ are used.

3. N-particle states and corresponding eigenvalues

Since $A(\lambda)$ generates the conserved quantities, for finding the physical observables we construct N-bosonic particle states as the eigenstates of the operator $A(\lambda)$ in the form

$$|\Phi_N(\mu_1 \ldots \mu_N)\rangle = \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_N)|0\rangle. \tag{21}$$

Since our Lax operator (4) in terms of the local quantum field q has a similar structure to the Lax operator of [10], the same line of argument as in [10] can be followed to show the existence of a vacuum state $|0\rangle$ satisfying the condition $B(\lambda)|0\rangle = 0$ and $A(\lambda)|0\rangle = |0\rangle$.

The eigenvalues of $A(\lambda)$ acting on the state $|\Phi_N\rangle$ may be obtained using the commutation relation (20c), which leads to

$$A(\lambda)|\Phi_N\rangle = a_N(\lambda, \boldsymbol{\mu})|\Phi_N\rangle$$

with

$$a_N(\lambda, \boldsymbol{\mu}) = \prod_{j=1}^{N} \frac{[\lambda - \exp(\mathrm{i}\gamma'/2)\mu_j][\lambda + \exp(-\mathrm{i}\gamma'/2)\mu_j]}{[\lambda - \exp(-\mathrm{i}\gamma'/2)\mu_j][\lambda + \exp(\mathrm{i}\gamma'/2)\mu_j]} \tag{22a}$$

where γ is replaced by $\tan(\gamma'/2)$ and a shift in the spectral parameter $\lambda \to \lambda \exp(-\mathrm{i}\gamma'/2)$ is implemented. A similar type of spectral shift is also needed earlier in other contexts [14] to remove asymmetry in the position of zeros and poles of a_N. Such formal shifting of parameters is required to neutralise the asymmetry appearing as a result of normal ordering [15]. To explore some physical interpretations we observe that, due to the Lorentz invariance of the system, we have $\phi = x^\mu K_\mu =$ invariant, with K_μ given by (16). Consequently $\lambda \to \lambda \exp(-\mathrm{i}\gamma'/2)$ is equivalent to the transition to a new Lorentz frame (x'^μ) with imaginary velocity $(-\mathrm{i}\gamma'/2)$. Note also that, under the above Lorentz transformation, masses remain invariant, while the complex-valued energy and momentum acquire real values, which is the motivation of our particular choice of the parameter. In terms of rapidities $a_N(\lambda, \boldsymbol{\mu})$ becomes

$$a_N(\lambda, \boldsymbol{\mu}) = \prod_{j=1}^{N} \tanh\left(\frac{\alpha - \beta_j + \mathrm{i}\gamma'/2}{2}\right) \coth\left(\frac{\alpha - \beta_j - \mathrm{i}\gamma'/2}{2}\right) \tag{22b}$$

where we have introduced $\lambda = \exp(\alpha)$ and $\mu_j = \exp(\beta_j)$.

The conserved quantities are determined by the eigenvalues of the asymptotic expansions of

$$\ln A(\lambda) = \begin{cases} \sum_{n=1}^{\infty} c_n \lambda^{-n} & \text{as} \quad \lambda \to \infty \\ \sum_{n=0}^{\infty} c_{-n} \lambda^{n} & \text{as} \quad \lambda \to 0. \end{cases} \tag{23}$$

In particular, the energy and momentum may be given by

$$H = \frac{\mathrm{i}m}{4|\gamma'|}(c_1 - c_{-1}) \qquad P = \frac{\mathrm{i}m}{4|\gamma'|}(c_1 + c_{-1}). \tag{24}$$

Therefore, the invariant mass of a scattering N-bosonic particle state yields

$$M_N = +(H_N^2 - P_N^2)^{1/2} = \frac{2m}{|\gamma'|}|\sin(\gamma'/2)|\left[\left(\sum_{j=1}^{N} \cosh\beta_j\right)^2 - \left(\sum_{j=1}^{N} \sinh\beta_j\right)^2\right]^{1/2}. \tag{25}$$

A distribution of rapidities of the particles in the form

$$\beta_j = \beta/N + \mathrm{i}\gamma'[j-(N+1)/2] \tag{26}$$

where β is real, cancels all zeros and poles from consecutive factors of $a_N(\lambda, \boldsymbol{\mu})$ leaving only two symmetrically placed zeros (poles) in the upper (lower) half λ plane, leading to

$$a_N^B(\lambda, \mu_0) = \frac{(\lambda - \mu_N)(\lambda + \bar{\mu}_N)}{(\lambda - \bar{\mu}_N)(\lambda + \mu_N)}$$

where

$$\mu_N = \mu_0 \exp(\mathrm{i}N\gamma'/2) \qquad \mu_0 = \exp(\beta/N) \tag{27}$$

and corresponds to a bound state

$$\mathscr{B}_N^\dagger(\mu_0)|0\rangle = \prod_{j=1}^N \bar{B}(\mu_j)|0\rangle \tag{28}$$

with analytic prolongation of μ_j as (26) and $\mathscr{B}_N^\dagger(\mu_0)$ as the N-particle breather 'creation' operator. The corresponding energy, momentum and mass are given by

$$H_N^{\mathrm{B}} = \frac{2m}{|\gamma'|} \sin\frac{N\gamma'}{2} \cosh\frac{\beta}{N} \qquad P_N^B = \frac{2m}{|\gamma'|} \sin\frac{N\gamma'}{2} \sinh\frac{\beta}{N} \tag{29a}$$

and

$$M_N^{\mathrm{B}} = +[(H_N^{\mathrm{B}})^2 - (P_N^{\mathrm{B}})^2]^{1/2} = \frac{2m}{|\gamma'|}\left|\sin\frac{N\gamma'}{2}\right| \tag{29b}$$

respectively. Consequently, the binding energy (mass) is $\Delta M_N^{\mathrm{B}} = NM_1^{\mathrm{B}} - M_N^{\mathrm{B}} \geqslant 0$ showing the breather to be a genuine bound state for $N > 1$. Note that for $N = N_{\max} = \pi/|\gamma'|$ we have $\mu_{N_{\max}} = \mathrm{i}\mu_0$ in (27) and consequently two zeros (poles) of a_N^{B} in the upper (lower) half λ plane merge together leading to

$$a_{N_{\max}}^{\mathrm{B}}(\lambda, \mu_0) = \left(\frac{\lambda - \mathrm{i}\mu_0}{\lambda + \mathrm{i}\mu_0}\right)^2$$

i.e. they yield a degenerate single zero (pole) on the imaginary axis, which clearly corresponds to the generation of soliton and antisoliton states from a breather state at this limit. One observes also that at this same limit $N \to N_{\max}$ the breather mass (29b) becomes equal to the mass of two solitons, where the soliton mass is given by $m/|\gamma'|$ as shown below. Therefore, (29b) is valid for $1 \leqslant N < N_{\max}$, which is also consistent with the constraint on γ.

Now in an attempt to find out consistent soliton states, we propose the following representation of breather operators:

$$\mathscr{B}_N^\dagger(\mu_0) = S^\dagger(\mu_0 \exp(\mathrm{i}N\theta)) S^\dagger(\mu_0 \exp(-\mathrm{i}N\theta)) \qquad \theta = \tfrac{1}{2}(\gamma' - \pi) \tag{30}$$

with commutation relation

$$A(\lambda) S^\dagger(\mu_0) = a^{\mathrm{S}}(\lambda, \mu_0) S^\dagger(\mu_0) A(\lambda) \qquad a^{\mathrm{S}}(\lambda, \mu_0) = \frac{\lambda - \mathrm{i}\mu_0}{\lambda + \mathrm{i}\mu_0} \tag{31}$$

for $\lambda \neq \mu_0$. Relation (30) for $N = 1$ yields the representation of an elementary bosonic creation operator through interacting soliton and antisoliton operators. The arguments of $S^{\dagger}$ in (30) are complex, indicating that the constituent particles do not have a separate existence, whereas in (31) $S^{\dagger}$ may create a free particle. However, at $N = N_{\max}$ one finds that $\mathscr{B}^{\dagger}_{N_{\max}}(\mu_0) = S^{\dagger}(\mu_0)S^{\dagger}(\mu_0)$ showing that at this limit the breather creation operator reduces to the creation of scattering soliton and antisoliton as also has been observed above. We intend to explore the properties of solitonic S operators in a forthcoming publication. The state with N scattering particles is given by $|\Psi_N(\mu_1 \ldots \mu_N)\rangle = S^{\dagger}(\mu_1) \ldots S^{\dagger}(\mu_N)|0\rangle$ with $a^{\mathrm{S}}_N(\lambda, \boldsymbol{\mu}) = \Pi^{N}_{j=1}(\lambda - \mathrm{i}\mu_j)/(\lambda + \mathrm{i}\mu_j)$ and the rest mass $M_N = Nm/|\gamma'|$. In analogy with the N-particle breather state (28), if we try to construct a soliton as an N-particle bound state assuming a distribution $\beta_j = \beta/N + \mathrm{i}\pi[j-(N+1)/2]$, we get

$$a^{\mathrm{S}}_N(\lambda, \mu_0) = \frac{\lambda - \mu_0 \exp(\mathrm{i}N\pi/2)}{\lambda - \mu_0 \exp(-\mathrm{i}N\pi/2)}$$

with a single zero (pole) on the imaginary λ axis, leading to a mass

$$M^{\mathrm{S}}_N = \frac{m}{|\gamma'|}\left|\sin\frac{N\pi}{2}\right| = m/|\gamma'|$$

for odd N and vanishing mass for even N. This may be interpreted as the even number of soliton-type particles annihilating each other leaving only a single soliton with mass $m/|\gamma'|$. It is fascinating that the masses of soliton and breather, obtained by us through an exact treatment, coincide with those found by Dashen *et al* [13] by a semiclassical approximation except for the difference that in our case $\gamma' = 2\tan^{-1}(\hbar\beta^2/2)$, while in the semiclassical case

$$\gamma' = \left(\frac{\hbar\beta^2}{8}\right)\left(1 - \frac{\hbar\beta^2}{8\pi}\right)^{-1}.$$

We now observe that an alternative possibility of novel bound states of breathers and also of solitons may exist with higher topological number n if one considers a distribution $\beta_j = \beta/N + (\mathrm{i}\gamma'/n)[j-(N+1)/2]$ for breathers and $\beta_j = \beta/N + (\mathrm{i}\pi/n)[j-(N+1)/2]$ for solitons. It is easy to verify that such distributions lead to $2n$ symmetrically placed zeros (poles) of $a_N(\lambda, \mu_0)$ in the upper (lower) half of the λ plane for 'n-breather' states with mass

$$M^{n-B}_N = \frac{2m}{|\gamma'|}\sum_{j=1}^{n}\sin\left[\frac{\gamma'}{2}\left(1 + \frac{N-j}{n}\right)\right]$$

and similarly to n zeros (poles) of $a_N(\lambda, \mu_0)$ in the upper (lower) half of the λ plane yielding 'n-soliton' states with mass

$$M^{n-S}_N = \frac{m}{|\gamma'|}\sum_{j=1}^{n}\sin\left[\frac{\pi}{2}\left(1 + \frac{N-j}{n}\right)\right].$$

These bound states have masses higher than that of $n = 1$ states and they are possibly only intermediate states, which ultimately break up into a superposition of n numbers of 1 breathers or 1 solitons. Since we cannot construct the explicit N-particle wavefunction in configuration space, it is not possible at this stage to prove rigorously the existence of these bound states.

4. Gauge invariance of commutation algebra and mass spectra at the quantum level

It is known that classical non-linear integrable equations have different Lax operator representations, which are gauge equivalent. However, in the corresponding quantum case such gauge equivalence may break down, since the gauge transformation and the normal ordering do not necessarily always commute. Consequently, the mass spectrum and other results obtained through the QIST treatment starting from a certain Lax operator are valid, in general, only for that particular gauge choice. However, it is, fascinating to note that, for the SG model in LCC, the commutation algebra and the mass spectrum obtained in a particular gauge, i.e. with Lax operator (4) also remain the same for the choice of standard AKNS type Lax operator (3) even at the quantum level.

We find that at the classical level the Lax operator (4) is related to (3) through a gauge transformation

$$L_{\mathrm{AKNS}} \xrightarrow{h} L = hL_{\mathrm{AKNS}}h^{-1} - h\partial_x h^{-1} \tag{32}$$

with $h(x) = \exp(\frac{1}{4}\mathrm{i}q(x)\sigma_1)$. But in the quantum case, apart from the above classical part, some additional anomalous terms appear due to the normal ordering of the fields. In particular, the non-trivial contribution comes here only from the normal ordering of the terms

$$\mathrm{i}\sigma_1 {:}h^{-1}(x){:}(\partial_x q^{(-)}(x) + \pi(x)){:}h(x){:} \tag{33}$$

with ${:}h(x){:} = \exp(\frac{1}{4}\mathrm{i}q^{(+)}(x)\sigma_1)\exp(\frac{1}{4}\mathrm{i}q^{(-)}(x)\sigma_1)$ for $\beta^2 < 8\pi$.

Note that all other singularities which may arise in (32) over the classical parts cancel among themselves resulting in no extra contribution, besides the terms in (33). We observe that, fortunately, the divergent pieces arising in this case are c-number terms and as a consequence,under such a gauge transformation, Γ will change only trivially with the addition of c-number terms containing a $(\mathbb{1}\otimes\mathbb{1})$ generator. As is evident from (13) such changes in Γ have no effect on the R matrix, i.e. they keep it the same as (14).

In order to find the R matrix corresponding to the AKNS Lax operator (3) satisfying the integrability condition

$$\tilde{R}\tilde{T}_y^x(\lambda)\otimes\tilde{T}_y^x(\mu) = \tilde{T}_y^x(\mu)\otimes\tilde{T}_y^x(\lambda)\tilde{R} \tag{34}$$

we observe that $T_y^x(\lambda)$, defined in (7), is gauge related to $\tilde{T}_y^x(\lambda)$ as

$$\tilde{T}_y^x(\lambda) = h^{-1}(x)T_y^x(\lambda)h(y). \tag{35}$$

On the other hand, since we have the relation (8), one may find the explicit form of $\tilde{R}$ in terms of the R matrix using (34) and (35). With the substitution of (35), (34) becomes

$$\begin{aligned}&\tilde{R}h^{-1}(x)T_y^x(\lambda)h(y)\otimes h^{-1}(x)T_y^x(\mu)h(y)\\&\qquad = h^{-1}(x)T_y^x(\mu)h(y)\otimes h^{-1}(x)T_y^x(\lambda)h(y)\tilde{R}.\end{aligned} \tag{36}$$

In order to compare the above equation with (8), we have to rearrange (36), which in turn may pick up additional terms due to non-commutativity of the local fields. To take proper care of such non-commutativity we write, for example, the LHS of (36) as

$$\left(h^{-1}(x)P\exp\int_{x-\Delta}^{x}\mathrm{d}\xi\,L(\xi,\lambda)\otimes h^{-1}(x)P\exp\int_{x-\Delta}^{x}\mathrm{d}\zeta\,L(\zeta,\mu)\right)\left(T_{y+\Delta}^{x-\Delta}(\lambda)\otimes T_{y+\Delta}^{x-\Delta}(\mu)\right)$$
$$\times\left(P\exp\int_{y}^{y+\Delta}\mathrm{d}\xi\,L(\xi,\lambda)h(y)\otimes P\exp\int_{y}^{y+\Delta}\mathrm{d}\zeta\,L(\zeta,\mu)h(y)\right)$$

and using the relation

$$[\pi(x),\exp(-\tfrac{1}{4}\mathrm{i}q(y))]=2\mathrm{i}\gamma\exp(-\tfrac{1}{4}\mathrm{i}q(y))\delta(x-y)$$

we find that

$$\lim_{\Delta\to0}\left(P\exp\int_{x-\Delta}^{x}\mathrm{d}\xi\,L(\xi,\lambda)\otimes h^{-1}(x)\right)$$
$$=\lim_{\Delta\to0}\left(1+\int_{x-\Delta}^{x}\mathrm{d}\xi\,L(\xi,\lambda)+\frac{1}{2!}\int_{k-\Delta}^{k}\mathrm{d}\xi\int_{k-\Delta}^{\xi}\mathrm{d}\xi'\,L(\xi,\lambda)L(\xi',\lambda)+\ldots\right)\otimes h^{-1}(x)$$
$$=\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)(\mathbb{1}\otimes h^{-1}(x)).$$

Similarly we have

$$\lim_{\Delta\to0}\left(h(y)\otimes P\exp\int_{y}^{y+\Delta}\mathrm{d}\zeta\,L(\zeta,\mu)\right)=\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)(h(y)\otimes\mathbb{1}).$$

As a result of this (36) leads to

$$\tilde{R}\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)(h^{-1}(x)\otimes h^{-1}(x))T_y^x(\lambda)\otimes T_y^x(\mu)(h(y)\otimes h(y))\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)$$
$$=\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)(h^{-1}(x)\otimes h^{-1}(x))T_y^x(\mu)\otimes T_y^x(\lambda)(h(y)\otimes h(y))\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)\tilde{R}$$

yielding finally

$$\tilde{R}(x,\lambda,\mu)=\exp(-\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)(h^{-1}(x)\otimes h^{-1}(x))R(\lambda,\mu)(h(x)\otimes h(x))\exp(\tfrac{1}{2}\gamma\sigma_1\otimes\sigma_1)$$
$$=h^{-1}(x)\otimes h^{-1}(x)R(\lambda,\mu)h(x)\otimes h(x). \tag{37}$$

Note that the x dependence of $\tilde{R}$ (37) vanishes as $x\to\pm\infty$ ultimately giving the result $\tilde{R}_\pm=R_\pm$ (19) which clearly yields the same mass spectrum (29*b*) as found before.

5. Discussion and conclusions

We have solved the quantum sine-Gordon model in light cone coordinates by finding a suitable Lax operator with the ultralocal property. The well known AKNS type Lax operator is, in fact, gauge related to the ultralocal Lax operator. It is interesting to note that, though in general gauge equivalence is violated at the quantum level due to the appearance of anomalous terms, for this particular model under investigation all the physically relevant quantities remain invariant under the gauge transformation, even in the quantum region. Therefore, we could find the solution also with the non-ultralocal Lax operator, which is not possible with the direct use of the quantum inverse scattering method. On the basis of the results obtained, we may conclude that the SG model contains (anti-)soliton type 'elementary' particles with mass $m/|\gamma'|$ and topological charge ± 1. Their superposition may generate a charge $\pm N$ state, which is a scattering state with mass $Nm/|\gamma'|$. A soliton and an antisoliton form an 'elementary' bosonic particle of mass $(2m/|\gamma'|)|\sin(\gamma'/2)$, the N number of which may construct again a non-topological breather bound state of mass $(2m/|\gamma'|)|\sin(N\gamma'/2)|$ with $N < \pi/|\gamma'|$. Possibly, there also exist n-solitons and n-breathers as other loosely bound states.

In [13] the following conjectures were made based on the WKB result. (i) The WKB results are exact for the SG model. (ii) All bound states M_N^B are stable and decay amplitudes of the processes $M_N^B \to M_N^B$ are zero. (iii) There exists a 'democracy' for all arbitrary N with the $N = 1$ configuration having no special role. In the light of our exact solution we are now in a position to comment on the validity of the above conjectures. The coincidence of our exact result with the semiclassical one [13] (apart from a renormalisation of the coupling constant) confirms the first conjecture. The second conjecture is proved by the conservation of particle number N in such theories, which is again an artefact of complete integrability and the existence of infinite conservation laws of SG, even at the quantum level. In connection with the third conjecture we should note that the binding energy ΔM_N^B of the breather state is non-vanishing only for $N > 1$. Moreover, only at $N = 1$ do the scattering bosonic particle state (21) and the bound state (28) become identical, as is evident from the comparison of their mass spectra (25) and (29*b*). Therefore, the $N = 1$ configuration may truly be interpreted as the elementary boson state and thus indeed plays a distinguished role, contradicting conjecture (iii). In [13] it is also argued that at $N = N_{\max}$ the breather becomes unstable and decays into a soliton and an antisoliton. We also observe from (30) that at $N\gamma' = \pi$, θ becomes $k\pi$ (k being an integer), making the arguments of constituting soliton operators real; this means that at this limit they become operators of free solitons, into which the breather possibly breaks up. The expression (29*b*) also shows that in this case the breather mass $M_{N_{\max}}^B = 2m/|\gamma'| = 2M^S$. In the limit $\gamma' \to 0$ when the sine-Gordon equation is converted into a non-interacting Klein-Gordon equation with free-particle mass m, the non-topological breathers reduce into free particles, while topological solitons do not have such a smooth transition. The topological soliton operator presented here possibly has fermionic properties and is related to the creation operators of the massive Thirring model. This problem is under investigation.

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